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MA 425/525
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## Homework 5

Due October 16th on paper at the beginning of class. Justify your answers. Please let me know if you have a question or find a mistake. The book is `https://archive.org/details/complex-variables-2ed-dover-1999-fisher/page/n23/mode/2up`.

Do 2.3.2, 2.3.6, 2.3.10 (but for this one replace $6+2i$ by $1+4i$ to make the calculation nicer and simplify your answer), 2.3.13, 2.3.14, 2.3.15, 2.3.16 from pages 116 and 117, 2.4.5 from page 133, 2.5.7 from page 150, 2.6.3 (for this one, you may assume $a \neq b$; note that the case $a = b$ follows by continuity) and 2.6.9 from page 167.

Also do the following:

1. Let $f$ be a function such that $f(z) = (z-p)^m g(z)$, where $m$ is an integer and $g$ is an analytic function with $g(p) \neq 0$. Let $h(z) = f'(z)/f(z)$. Compute $\operatorname{Res}(h;p)$.
2. Let
$$f(z) = \frac{\sin\frac{z}{2}\sin\frac{z}{3}}{\sin\frac{z}{5}\sin\frac{z}{7}}.$$
Find a point where $f$ has a simple pole, a point where it has a double pole, a point where it has a simple zero, a point where it has a double zero, and a point where it has a removable singularity.

   Finally, you may like to (but are not required to) examine $f$ near these points using `https://samuelj.li/complex-function-plotter`

*Hints:* For 2.3.10, note that the difference of Args simplifies, even though neither Arg alone does. For 2.3.13, there is a hint below Figure 2.8. [1] For 2.3.14, take the usual parametrization of the circle (as in Example 2 on page 57), plug into the Cauchy formula, and simplify. For 2.3.15 and 2.3.16, recall that if a function $g$ has a strict local maximum at $p$ then there is $\varepsilon > 0$ such that $g(z) < g(p)$ whenever $0 < |z-p| \leq \varepsilon$, and similarly for a minimum. For 2.4.5 use a half angle formula. For 2.6.9, also look at Example 6 from page 112 and use (7) from page 138 to compute the residue. For 1, use the fact that $g'(z)/g(z)$ is analytic near $p$. [2]

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[1] You are not required to verify that $\sin x \geq 2x/\pi$ for $x \in [0, \pi/2]$, but in case you are curious, one way to do this is to let $g(x) = \sin x - 2x/\pi$, and show that 1) $g(0) = g(\pi/2) = 0$, 2) $g'(0) > 0$, 3) $g'$ has only one root in $[0, \pi/2]$.

[2] Incidentally, this means that if $\Gamma$ is a contour enclosing a single zero or pole of $f$, then $\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz$ computes the order of that zero or pole. This will be important in Chapter 3 when we analyze zeroes and poles more closely.